Trastos cruzados en Topología Algebraica no abeliana

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Michael Atiyah
Mathematics in the 20th century
Bull. London Math. Soc. **34** (2002), 1–15.

1. Local to global
2. Increase in dimensions
3. Commutative to non-commutative
4. Geometry versus algebra
5. Techniques in common
   - Homology theory
   - K-theory
6. Impact of physics

We study a part of algebraic topology which lies between homology theory and homotopy theory, and in which the fundamental group and its actions plays an essential role.

Main applications are to higher dimensional nonabelian methods for local-to-global problems, as exemplified by van Kampen type theorems.

Some calculations seem at this stage of the subject to require strict algebraic models of homotopy types. In this process some nonabelian calculations are obtained, and it is this methodology which is called nonabelian algebraic topology.

R. Brown, P. J. Higgins and R. Sivera,
Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids
EMS Tracts in Mathematics, European Mathematical Society (EMS), Zürich, 2011.

It is fortunate that higher categorical structures do give nonabelian algebraic models of homotopy types which allow some explicit calculation.

They have also led to new algebraic constructions, such as a nonabelian tensor product of groups, of Lie algebras, and of other algebraic structures, with relations to homology of these structures.

Topologists of the early 20th century dreamed of a generalisation to higher dimensions of the nonabelian fundamental group, for applications to problems in geometry and analysis for which group theory had been successful.

- The nonabelian fundamental group $\pi_1(X, x_0)$ was important in analysis, geometry, and topology.
- Homology groups $H_n(X)$ existed in all dimensions, and were abelian.
- X connected implies $H_1(X) \cong \pi_1(X, x_0)_{\mathrm{ab}}$, the fundamental group made abelian.

So they dreamed of a higher dimensional generalisation of the fundamental group.

The dream seemed to be shattered by the discovery that Čech's apparently natural 1932, for the ICM at Zürich, generalisation of the fundamental group, the higher homotopy groups $\pi_n(X, x_0)$, were abelian in dimensions ≥ 2.

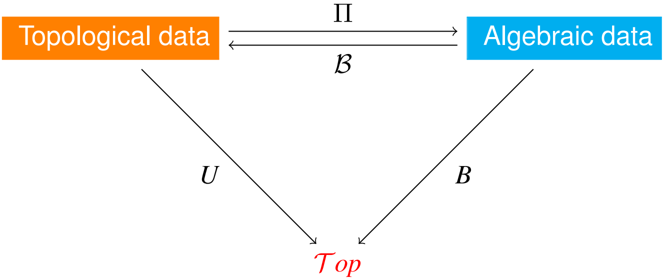
Alexandroff and Hopf proved these groups were abelian and persuaded Čech to withdraw his paper.

We now see this as group objects internal to the category of groups are just abelian groups.

Problem

To recapture the higher dimensional information.

Nonabelian Algebraic Topology



1. There is a natural equivalence $\Pi \circ \mathcal{B} \simeq 1$.
2. U is a forgetful functor and $B = U \circ \mathcal{B}$.
3. Π satisfies a van Kampen theorem (it preserves certain colimits).
4. There is a natural transformation $1 \to \mathcal{B} \circ \Pi$ with good properties (preserving some homotopy).
5. ideally: homotopy classification

$$[UX, BC] \cong [\Pi X, C]\,.$$

Precrossed and crossed modules

Definition

A precrossed module (M, P, μ) is:

- a group homomorphism $\mu \colon M \longrightarrow P$,
- with an action of the group P on M, denoted ${}^{p}m$, for $p \in P$ and $m \in M$, satisfying $\mu({}^{p}m) = p\mu(m)p^{-1}$.

Definition

(M, P, μ) is a crossed module if besides it satisfies the Peiffer's identity ${}^{\mu(m)}m' = mm'm^{-1}$ for every $m, m' \in M$.

J.H.C. Whitehead
Combinatorial homotopy II
Bull. Amer. Math. Soc. **55** (1949), 453–496.

Morphisms of precrossed and crossed modules

Definition

A morphism of precrossed (crossed) modules

$$(\Phi, \Psi)\colon (M_1, P_1, \mu_1) \to (M_2, P_2, \mu_2)$$

is a pair of group homomorphisms Φ and Ψ such that

$$\begin{array}{ccc} M_1 & \xrightarrow{\Phi} & M_2 \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle \mu_2} \\ P_1 & \xrightarrow{\Psi} & P_2 \end{array}$$

commutes, and such that $\Phi({}^{p_1}m_1) = {}^{\Psi(p_1)}\Phi(m_1)$ for every $m_1 \in M_1$ and $p_1 \in P_1$.

Notation.

- $\mathcal{PCM}$: Category of precrossed modules.
- $\mathcal{CM}$: Category of crossed modules.

Peiffer abelianisation functor

Definition

The Peiffer subgroup of a precrossed module (M, P, μ) is the subgroup $\langle M, M \rangle$ of M generated by the *Peiffer elements* $\langle m_1, m_2 \rangle = m_1 m_2 m_1^{-1}\, {}^{\mu(m_1)} m_2^{-1}$ with $m_1, m_2 \in M$.

Definition

The Peiffer abelianisation functor $\mathrm{Peiff}\colon \mathcal{PCM} \to \mathcal{CM}$ assigns to a precrossed module (M, P, μ) the quotient $(M, P, \mu)_{\mathrm{Peiff}} = (M, P, \mu)/(\langle M, M \rangle, 0, 0)$.

This crossed module is not abelian in general.

Standard examples of precrossed and crossed modules

Examples

If Y is a path-connected topological space, and X is obtained from Y attaching 2-cells, then the map $\partial\colon \pi_2(X,Y,x_0) \to \pi_1(Y,x_0)$ is a crossed module.

We have a functor $\Pi_2\colon \mathcal{T}op_*^2 \to \mathcal{CM}$

J.H.C. Whitehead
Combinatorial homotopy II
Bull. Amer. Math. Soc. **55** (1949), 453–496.

Standard examples of precrossed and crossed modules

Examples

- N normal subgroup of G: (N, G, i) is a crossed module, where $i\colon N \hookrightarrow G$ denotes the inclusion and G acts on N by conjugation (inclusion crossed module).
- P a G-group: $(P, G, 0)$ is a precrossed module, which is a crossed module if P is a G-module.
- In particular, a group G can be regarded as a crossed module (G, G, id), $(0, G, 0)$, or as a precrossed module $(G, 0, 0)$ which is a crossed module only if G is abelian.
- If G is a group, its automorphism crossed module is given by $(G, \mathrm{Aut}(G), \mu)$, where for $g \in G$, $\mu(g)$ is the inner automorphism of G mapping $x \mapsto gxg^{-1}$.

Standard examples of precrossed and crossed modules

Examples (Fundamental crossed module of a fibration)

Let $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} B)$ be a fibration of pointed spaces, with $F = p^{-1}(b_0)$ the fibre, where b_0 is the base point of B.

Then the induced map

$$\Pi_2(\mathcal{F}) = \pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E)$$

is a crossed module.

Standard examples of precrossed and crossed modules

Examples (Central extension crossed module)

Let $\partial\colon M \to G$ be a surjective morphism with $\ker\partial \subset \mathrm{Z}(M)$, and $g \in G$ acts on $m \in M$ by conjugation with any element of $\partial^{-1}(g)$.

$$0 \to \ker\partial \to M \to G \to 0$$

Then $\partial\colon M \to G$ is a crossed module.

Non-abelian tensor product

Definition (Brown & Loday, 1984)

Given two groups M and N equipped with an action of M on N and an action of N on M the nonabelian tensor product $M \otimes N$ is the group generated by the symbols $m \otimes n$, for $m \in M$ and $n \in N$, with relations

$$\begin{aligned} mm' \otimes n &= ({}^{m}m' \otimes {}^{m}n)(m \otimes n) \\ m \otimes nn' &= (m \otimes n)({}^{n}m \otimes {}^{n}n') \end{aligned}$$

for all $m, m' \in M$ and $n, n' \in N$, understanding that each group acts on itself by conjugation.

$M \otimes N$ is not abelian group.

In case that the actions of the groups on each other are trivial, but the groups are not necessarily abelian, then

$$M \otimes N \cong M_{\mathrm{ab}} \otimes N_{\mathrm{ab}} \,.$$

Standard examples of precrossed and crossed modules

Examples

Let M and N be normal subgroups of G.

The morphism

$$\begin{aligned} \partial\colon M \otimes N &\to G \\ m \otimes n &\mapsto mnm^{-1}n^{-1}\,, \end{aligned}$$

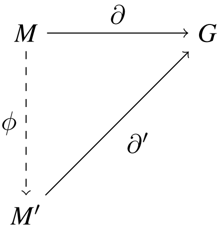
with the action ${}^{g}(m \otimes n) = gmg^{-1} \otimes gng^{-1}$ is a crossed module.

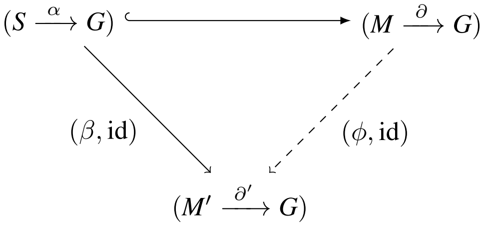
Particular case: $(G \otimes G, G, \partial)$.

Free crossed modules

$\partial\colon M \to G$ a crossed module is free on the function $\alpha\colon S \to G$ for some set S if:

- S is a subset of M,
- α is the restriction of ∂,
- and with the universal property that for any crossed module $\partial'\colon M' \to G$ and function $\beta\colon S \to M'$ such that $\partial'\beta = \alpha$, there is a unique morphism $\phi\colon M \to M'$ of crossed G-modules such that





Adjunction

The forgetful functor $\mathcal{U}\colon \mathcal{CM}/G \longrightarrow \mathcal{S}et/G$ has a left adjoint functor

$$\begin{array}{ccc} \mathcal{S}et/G & S \xrightarrow{\alpha} G & M \xrightarrow{\partial} G \\ \mathcal{F} \downarrow\uparrow \mathcal{U} & \mathcal{F}\downarrow & \uparrow \mathcal{U} \\ \mathcal{CM}/G & \mathcal{F}(S \xrightarrow{\alpha} G) & M \xrightarrow{\partial} G \end{array}$$

The free functor $\mathcal{F}\colon \mathcal{S}et/G \longrightarrow \mathcal{CM}/G$, is defined by

$$\mathcal{F}(S \xrightarrow{\alpha} G) = (F, G, \partial) ,$$

where F is the free group on the set $G \times S$ modulo Peiffer elements, where

- the action is $\quad g_1 \cdot (g, s) = (g_1 g, s)$,
- $\partial\colon F \to G$, $\qquad \partial(g, s) = g(\alpha s)g^{-1}$, $\quad$ and
- $(g_1, s_1)(g_2, s_2)(g_1, s_1)^{-1}\big((g_1 \alpha s_1 g_1^{-1}) \cdot (g_2, s_2)^{-1}\big) = 1$.

Examples of free crossed modules

Examples

- If Y is a path-connected topological space, and X is obtained from Y attaching 2-cells, then

$$\Pi_2(X,Y) = \pi_2(X,Y,x_0) \xrightarrow{\partial} \pi_1(Y,x_0)$$

  is a free crossed module.

  J.H.C. Whitehead
  Combinatorial homotopy II
  Bull. Amer. Math. Soc. **55** (1949), 453–496.

- If M is a free $\mathbb{Z}G$-module then $(M, G, 0)$ is a free crossed module.
- If F is a free group then (F, F, id) is a free crossed module.

Combinatorial Group Theory

- Let $\langle X, R\rangle$ a presentation of the group G, i.e., $F(X)/\langle R\rangle = G$.
- The free crossed module $\partial\colon C \to F(X)$ on the function $R \hookrightarrow F(X)$ satisfies:

  $\operatorname{im}\partial = \langle R\rangle$ and therefore $\operatorname{coker}\partial = G$.

$\ker\partial$ is a measure of the *no trivial identities among relations* and are known as the modulo of identities or nonabelian syzygies.

Characterization of free crossed modules

Theorem (J. G. Ratcliffe, 1980)

(M, G, μ) is a free crossed module with basis $\{m_s\}_{s\in S}$ if, and only if:

1. *M_{ab} is a free $G/\mu(M)$-module with basis $\{[m_s]\}_{s\in S}$;*
2. *$\mu(M)$ is the normal closure of $\{\mu(m_s)\}_{s\in S}$ in G ;*
3. *$\mu_* : H_2(M) \to H_2(G)$ is trivial.*

Cohomology of groups

The description of the second cohomology of a group in terms of extensions of groups

$$H^2(G,A): \quad [0 \to A \to E \to G \to 1]$$

led to a desire to find analogous interpretations of the third and higher cohomology groups.

The third cohomology of a group of Eilenberg-Mac Lane $H^3(G,A)$ is described in terms of crossed 2-fold extensions of the G-module A by the group G,

$$0 \to A \to X \to Y \to G \to 1$$

where $X \to Y$ is a crossed module.

(Co)homology of groups

Huebschmann, 1980

Internal approach of the theory of (Co)homology of groups:

If A is a Q-module, then $H^n(Q,A)$, $H_n(Q,A)$, the n-th groups of cohomology and homology can be obtained from a crossed resolution:

$$\mathbb{T}\colon\ \cdots T_{k+1} \to T_k \to \cdots \to T_2 \to T_1 \to F \to Q \to 1,$$

where $T_1 \to F$ is a free crossed module and T_n are free Q-modules.

Epimorphisms

Characterization of the epimorphisms in the category of crossed modules

A morphism $(\phi, \eta)\colon (T', G', \partial') \to (T, G, \partial)$ is an epimorphism if, and only if,

1. η is surjective,
2. $\phi(T')$ is a normal subgroup of T, and
3. $T/\phi(T')$ is a perfect group.

If G is a perfect group, then

$$(0, \mathrm{id})\colon (0, G, 0) \to (G, G, \mathrm{id})$$

is an epimorphism which is not surjective.

Categorification

The term categorification was introduced by Crane.

Louis Crane
Clock and category: Is quantum gravity algebraic?
J. Math. Phys. **36** (1995), 6180–6193.

The term refers to the process of replacing set-theoretic notions by the corresponding category-theoretic analogues.

Categorification

Set Theory	Category Theory
set	category
element	object
relation between elements	morphism of objects
function	functor
relation between functions	natural transformation of functors

The general idea is that, replacing a *simpler* object by something *more complicated*, one gets a bonus in the form of some extra structure which may be used to study the original object.

Categorification

Example

The category of finite sets may be considered as a categorification of the semi-ring $(\mathbb{N}_0, +, \cdot)$ of non-negative integers.

In this picture addition is categorified via the disjoint union and multiplication via the cartesian product.

Decategorification

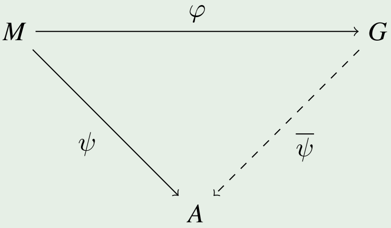
It is always easier to *forget* information than to *make it up*. Therefore it is much more natural to start the study of categorification with the study of the opposite process of forgetting information, called decategorification.

Decategorification

Example (Grothendieck group)

Originally, the Grothendieck group was defined for a commutative monoid and provided a universal way of making that monoid into an abelian group.

Let $M = (M, +, 0)$ be a commutative monoid. The Grothendieck group of M is a pair (G, φ), where G is a commutative group and $\varphi\colon M \to G$ is a homomorphism of monoids, and with the universal property



where A is a commutative group.

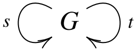
Categorification

J. C. Baez and A. D. Lauda
Higher-dimensional algebra. V. 2-groups
Theory Appl. Categ. **12** (2004), 423–491.

J. C. Baez and A. S. Crans
Higher-dimensional algebra. VI. Lie 2-algebras
Theory Appl. Categ. **12** (2004), 492–538.

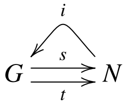
Equivalent notions of precrossed module

1. Internal reflexive graph in $\mathcal{G}roups$ (or group-graph):



   where s and t satisfy $st = t$ and $ts = s$.

2. Pre-cat^1-group:



   where $si = \mathrm{id} = ti$.

3. Group object in $\mathcal{G}raphs$.

Equivalent notions of crossed module

1. Internal category in $\mathcal{G}r$ (or strict 2-group):
   $$C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0 \xrightarrow{i} C_1$$
   where s, t and i satisfy $si = ti = \mathrm{id}_{C_0}$ and a homomorphism $C_1 \times_{s,t} C_1 \xrightarrow{\text{comp}} C_1$.

2. Cat^1-group:
   $$G \underset{t}{\overset{s}{\rightrightarrows}} G$$
   - $st = t, \quad ts = s,$
   - $[\ker(s), \ker(t)] = 0.$

3. Group object in $\mathcal{C}at$:
   It is an object G in the category and arrows $1\colon * \to G$, the unit map, $(-)^{-1}\colon G \to G$, the inverse map, and $m\colon G \times G \to G$, the multiplication map, satisfying the axioms of group.

Embedding

The category of groups, $\mathcal{G}r$, can be considered a full subcategory of $\mathcal{C}at^1 - \mathcal{G}r$ using the inclusion functor:

$$I\colon \mathcal{G}r \to \mathcal{C}at^1 - \mathcal{G}r$$

given by $I(G) = (G, \mathrm{id}_G, \mathrm{id}_G)$.

Equivalence of categories

We have a functor

$$\Phi\colon \mathcal{CM} \to \mathcal{C}at^1 - \mathcal{G}r$$

$$\Phi(M, P, \mu) = (M \rtimes P, s, t)\,,$$

where $s(m, p) = (1, p)$ and $t(m, p) = (1, \mu(m)p)$.

Inversely, we have a functor

$$\Psi\colon \mathcal{C}at^1 - \mathcal{G}r \to \mathcal{CM}$$

$$\Psi(G, s, t) = \left(t_{|\ker(s)}\colon\ \ker s \to \mathrm{im}(s)\right)\,,$$

where $\mathrm{im}\, s$ acts on $\ker s$ by conjugation.

We have an equivalence of categories $\mathcal{C}at^1 - \mathcal{G}r \to \mathcal{CM}$.

Identification

Therefore, the natural identification of a group as crossed module is

$$G \equiv (0, G, 0)\,.$$

Classifying space

The classifying space of a group P is a functorial construction

$$B\colon \mathcal{G}r \to \mathcal{T}op_*$$

assigning a reduced CW-complex BP to each group P so that

the homotopy groups of the classifying space BP satisfy

$$\pi_i(BP) \cong \begin{cases} P & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i \geq 2. \end{cases}$$

Classifying space

If X is a reduced CW-complex then there is a map $X \to B\pi_1(X)$ inducing an isomorphism of fundamental groups, i.e.,

there is a natural transformation $1 \to B\pi$ preserving some homotopy properties.

Moreover, there is natural equivalence $\pi B \simeq 1$.

$$\begin{array}{c} \mathcal{T}op_* \\ \pi \downarrow\uparrow B \\ \mathcal{G}r \end{array}$$

Algebraic models

The idea of a homotopy n-type is that of a space for which all the homotopy groups of order higher than n are trivial.

The groups are algebraic models of homotopy 1-types.

Classifying space

The classifying space of a crossed module (M, P, μ) is a functorial construction

$$\mathcal{B}\colon \mathcal{CM} \to \mathcal{T}op_*$$

assigning a pointed CW-space $\mathcal{B}(M, P, \mu)$ to each crossed module (M, P, μ) with the following properties:

the homotopy groups of the classifying space $\mathcal{B}(M, P, \mu)$ satisfy

$$\pi_i \mathcal{B}(M, P, \mu) \cong \begin{cases} \operatorname{coker}\mu & \text{if} \quad i = 1, \\ \ker\mu & \text{if} \quad i = 2, \\ 0 & \text{if} \quad i \geq 3. \end{cases}$$

Classifying space

If P is a group then $BP = \mathcal{B}(0, P, 0)$.

The classifying space BP is a subcomplex of $\mathcal{B}(M, P, \mu)$, and there is a natural isomorphism of crossed modules

$$\Pi_2(\mathcal{B}(M, P, \mu), BP) \cong (M, P, \mu)\,.$$

We have a functor

$$\begin{aligned} \mathcal{B}\colon \mathcal{CM} &\to \mathcal{T}op^2_* \\ (M, P, \mu) &\mapsto (\mathcal{B}(M, P, \mu), BP) \end{aligned}$$

Classifying space

Let X be a reduced CW-complex, and let $\Pi_2(X, X^1)$ be the crossed module $\pi_2(X, X^1) \to \pi_1(X^1)$, where X^1 is the 1-skeleton of X.

Then there is a map $X \to \mathcal{B}\Pi_2(X, X^1)$ inducing an isomorphism of π_1 and π_2, i.e.,

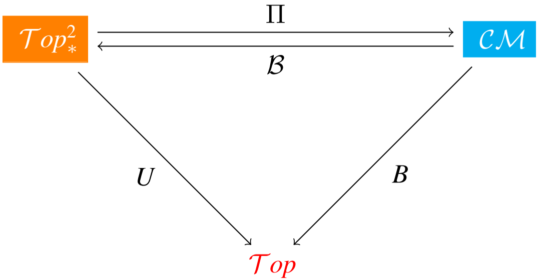
there is a natural transformation $1 \to \mathcal{B}\Pi$ preserving some homotopy properties.

Moreover, there is a natural equivalence $\Pi\mathcal{B} \simeq 1$.

Algebraic models

The crossed modules are algebraic models of homotopy 2-types.

$$\mathcal{T}op^2_* \\ \Pi \downarrow\uparrow \mathcal{B} \\ \mathcal{CM}$$



local-global

Let A, U_1, U_2 be subspaces of X such that the connected space X is the union of the interior of two connected subspaces U_1 y U_2, with connected intersection $U_{12} = U_1 \cap U_2$ and $A_\nu = A \cap U_\nu$, $\nu = 1, 2, 12$.

Then the following diagram induced by the inclusions is a pushout of crossed modules

$$\begin{array}{ccc} \Pi_2(U_{12}, A_{12}) & \longrightarrow & \Pi_2(U_2, A_2) \\ \downarrow & & \downarrow \\ \Pi_2(U_1, A_1) & \longrightarrow & \Pi_2(X, A). \end{array}$$

(Co)Homology

A theory for the (co)homology of crossed modules was introduced by G. J. Ellis (1992), via classifying spaces.

G. J. Ellis
Homology of 2-types
Journal of the London Mathematical Society. Second Series **46** (1992), 1–27.

Let (M, P, μ) be a crossed module and A a π_1-module, where $\pi_1 = P/\mu(M)$. The homology and cohomology of (M, P, μ) with coefficients in A are defined by

$$H_n\big((M, P, \mu), A\big) = H_n(\mathcal{B}(M, P, \mu), A)$$

$$H^n\big((M, P, \mu), A\big) = H^n(\mathcal{B}(M, P, \mu), A)$$

Algebraic category

The category of crossed modules is algebraic over sets.

$$\begin{array}{c} \mathcal{S}et \\ \mathcal{F} \downarrow\uparrow \mathcal{U} \\ \mathcal{CM} \end{array}$$

$$\begin{array}{c} \mathcal{S}et \\ \mathcal{F}_1 \downarrow\uparrow \mathcal{U}_1 \\ \mathcal{G}r \\ \mathcal{F}_2 \downarrow\uparrow \mathcal{U}_2 \\ \mathcal{CM} \end{array}$$

Algebraic category

The functor $\mathcal{U}_2$ is defined by:

$$\mathcal{U}_2(M, P, \mu) = M \times P$$

The functor $\mathcal{F}_2$ is defined by:

$$\mathcal{F}_2(G) = (\overline{G}, G * G, \mathrm{inc})$$

where $\overline{G} = \ker(G * G \xrightarrow{p_2} G)$.

Therefore, the functor $\mathcal{F}$ is:

$$\mathcal{F}(X) = (\overline{F(X)}, F(X) * F(X), \mathrm{inc}).$$

Homology of crossed modules

The abelianisation of a crossed module $\partial\colon T \to G$ is the morphisms of abelian groups

$$\mathrm{ab}(\partial)\colon T/[G,T] \to G/[G,G].$$

The category of abelian crossed modules is equivalent to the category of right modules over the ring of matrices

$$\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

Homology of crossed modules

The CCG-homology of crossed modules was introduced by P. Carrasco, A. M. Cegarra and A. R.-Grandjeán:

P. Carrasco, A. M. Cegarra and A. R.-Grandjeán
(Co)homology of crossed modules
J. Pure Appl. Algebra **168** (2002), 147–176.

$$H_n(M,P,\mu) = H_{n-1}\big((M_\cdot, P_\cdot, \mu_\cdot)_{ab}, \partial_*\big)$$

$$H^n\big((M,P,\mu),(A,B,f)\big) = H^{n-1}\big(\operatorname{Hom}_{\mathcal{PCM}}\big((M_\cdot,P_\cdot,\mu_\cdot),(A,B,f)\big),\partial^*\big)$$

where $(M,P,\mu)_*$ is a projective presentation of (M,P,μ).

Homology of crossed modules

It is proved that for $(0,G,0)$, the homology agrees with the integral homology of G,

$$H_n(0,G,0) = H_n(G)\,.$$

$$H^n\big((0,G,0),(A,B,f)\big) = H^n(G,B)\,,$$

are the cohomology groups of Eilenberg-Mac Lane.

Homology of crossed modules

$H_i^{\mathrm{CCG}}(T, G, \partial)$ is a homomorphism of abelian groups

$$\zeta H_i^{\mathrm{CCG}}(T, G, \partial) \to \kappa H_i^{\mathrm{CCG}}(T, G, \partial).$$

The group $\kappa H_i^{\mathrm{CCG}}(T, G, \partial)$ is the integral homology $H_i(BG)$ of the classifying space BG.

We prove that

$$\zeta H_i^{\mathrm{CCG}}(T, G, \partial) = H_{i+1}\beta(T, G, \partial)$$

where $\beta(T, G, \partial)$ is the cofibre of the canonical map

$$i \colon BG \to \mathcal{B}(T, G, \partial).$$

A. R.-Grandjeán, M. Ladra and T. Pirashvili
CCG-homology of crossed modules via classifying spaces
J. Algebra **229** (2000), 660–665.

Homology of crossed modules

Therefore, the CCG-homology is closely related to the homology of crossed modules, defined by G. J. Ellis, via classifying spaces.

Given a G-module M, we have the isomorphisms:

$$\zeta H_1^{\mathrm{CCG}}(M, G, 0) = H_0(G, M)$$

$$\zeta H_2^{\mathrm{CCG}}(M, G, 0) = H_1(G, M)$$

Example from algebraic K-theory

Let R be an associative ring with unit.

- $E(R)$ is the subgroup of $GL(R)$ generated by the elementary matrices.
- $St(R)$ is the Steinberg group, generated by elements $x_{ij}(r)$, with $i \neq j \in \mathbb{Z}$ and $r \in R$, subject to the relations

$$\begin{aligned} x_{ij}(r)x_{ij}(s) &= x_{ij}(r+s), \\ [x_{ij}(r), x_{kl}(s)] &= 1 \text{ if } j \neq k \text{ and } i \neq l, \\ [x_{ij}(r), x_{jk}(s)] &= x_{ik}(rs) \text{ if } i \neq k. \end{aligned}$$

Example from algebraic K-theory

- $K_1(R)$ is the first algebraic K-theory group of R.
- $K_2(R)$ is the second algebraic K-theory group of R.

Example

The nonabelian tensor product $E(R) \otimes E(R) \cong St(R)$ and therefore $\partial\colon St(R) \to GL(R)$ is a crossed module.

Moreover, $\operatorname{coker} \partial = K_1(R)$ and $\ker \partial = K_2(R)$.

Example from algebraic K-theory

For a two-sided ideal I of a ring R there exists a perfect crossed module $(E(I), E(R), i)$, where

- $E(I) = E(R) \cap GL(I)$.

Theorem (Gilbert, 2000)

The universal central extension of $(E(I), E(R), i)$ *in* $\mathcal{CM}$ *is*

$$(K_2(R,I), K_2(R), \overline{\gamma}) \rightarrowtail (St(R,I), St(R), \overline{\gamma}) \twoheadrightarrow (E(I), E(R), i)$$

where

- $St(R,I)$ *is the relative Steinberg group defined by Loday and Keune.*
- $K_2(R,I)$ *denotes the second relative K-theory group introduced by Loday and Keune.*

Example from algebraic K-theory

Theorem

The universal central extension in $\mathcal{PCM}$ of $(E(I), E(R), i)$ is

$$(K_2(I), K_2(R), \gamma) \rightarrowtail (St(I), St(R), \gamma) \twoheadrightarrow (E(I), E(R), i),$$

where

- $St(I)$ *and* $K_2(I)$ *denote their Stein relativizations.*

D. Arias, M. Ladra and A. R.-Grandjeán
Universal central extensions of precrossed modules and Milnor's relative K_2
J. Pure Appl. Algebra **184** (2003), 139–154.

Example from algebraic K-theory

Given a two-sided ideal I of a ring R and a functor $\Phi\colon \mathcal{R}ings \to \mathcal{G}r$, the Stein relative group $\Phi(I)$ can be defined as follows:
denote by D the pullback of the natural ring homomorphism $R \twoheadrightarrow R/I$.

$$\begin{array}{ccc} D & \xrightarrow{p_1} & R \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ R & \longrightarrow & R/I \end{array}$$

The projections p_1 and p_2 are split by the diagonal homomorphism $\Delta\colon R \to D$.
These ring homomorphisms induce the group homomorphisms

$$\Phi(D) \underset{p_{2*}}{\overset{\overset{\Delta_*}{\curvearrowleft}\;\; p_{1*}}{\rightrightarrows}} \Phi(R)$$

$\Phi(I)$ is defined as the kernel of p_{1*}.

Example from algebraic K-theory

$$
\begin{array}{ccccc}
H_2^{PCM}(St(R,I),St(R),\overline{\gamma}) & = & H_2^{PCM}(St(R,I),St(R),\overline{\gamma}) & & \\
\big\downarrow\text{(mono)} & & \big\downarrow\text{(mono)} & & \\
H_2^{PCM}(E(I),E(R),i) & \rightarrowtail & (St(I),St(R),\gamma) & \twoheadrightarrow & (E(I),E(R),i) \\
\big\downarrow\text{(epi)} & & \big\downarrow\text{(epi)} & & \big\| \\
H_2^{CM}(E(I),E(R),i) & \rightarrowtail & (St(R,I),St(R),\overline{\gamma}) & \twoheadrightarrow & (E(I),E(R),i)
\end{array}
$$

Example from algebraic K-theory

- $$H_2^{PCM}(E(I), E(R), i) \cong (K_2(I), K_2(R), \gamma)$$

- $$H_2^{CM}(E(I), E(R), i) \cong (K_2(R, I), K_2(R), \overline{\gamma})$$

- $$(St(R, I), St(R), \overline{\gamma}) = (St(I), St(R), \gamma)_{\text{Peiff}}$$

- $$K_2(R, I) \cong \frac{K_2(I)}{\langle St(I), St(I) \rangle}$$

GRACIÑAS!